

Queen Mary, University of London

Examination by Course Unit

MAS317 Linear Algebra II

Duration: 2 hours

Date and time: 4 June 2007, 2.30 pm

*You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best **FOUR** questions answered will be counted.*

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

The symbol i denotes the complex number $\sqrt{-1}$ throughout the paper.

You are not allowed to start reading the question paper until instructed to do so by the invigilator.

You must not remove the examination paper from the examination room.

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Question 1 (a) Let A and B be two $n \times n$ complex matrices. Explain what is meant by saying that A is *congruent* to B . What is meant by saying that A is *unitary*?

Given that $A = (a_{jk})$ is a unitary matrix, show that AB is congruent to $B\bar{A}$, where $\bar{A} = (\bar{a}_{jk})$ and $\bar{a}_{jk}$ is the complex conjugate of a_{jk} .

(b) Determine with reasons which of the following matrices are congruent to each other over $\mathbb{R}$:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}.$$

Further, find all real values of x such that the matrix $\begin{pmatrix} x & 1 \\ 1 & 1 \end{pmatrix}$ is congruent to C .

(c) Given a complex symmetric matrix

$$A = \begin{pmatrix} 0 & i & 1 \\ i & 0 & -i \\ 1 & -i & 0 \end{pmatrix},$$

find a matrix P such that $P^T A P$ is a diagonal matrix, where P^T denotes the transpose of P .

Question 2 Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be the linear map given by

$$f(t, x, y, z) = (x + z, t + y, x + z, t + y).$$

- (i) Find the matrix $(f; \mathcal{B}, \mathcal{B})$ representing f with respect to the standard basis $\mathcal{B}$ for both the domain $\mathbb{R}^4$ and co-domain $\mathbb{R}^4$.
- (ii) Find a basis for the kernel of f .
- (iii) Find the eigenvalues of $(f; \mathcal{B}, \mathcal{B})$.
- (iv) Find a basis $\mathcal{C}$ of $\mathbb{R}^4$, consisting of eigenvectors of $(f; \mathcal{B}, \mathcal{B})$, such that the representing matrix $(f; \mathcal{C}, \mathcal{C})$ of f is a diagonal matrix.

Question 3 (a) Let V be a vector space over a field F and let f be a bilinear form on V . Explain what is meant by saying that f is *symmetric*.

Let $q : V \longrightarrow F$ be a quadratic form given by

$$q(v) = f(v, v) \quad (v \in V)$$

where f is a symmetric bilinear form. Show that

$$2f(u, v) = q(u + v) - q(u) - q(v) \quad (u, v \in V).$$

Find a symmetric bilinear form f on $\mathbb{R}^2$ such that

$$f((x, y), (x, y)) = x^2 - xy + y^2 \quad (x, y \in \mathbb{R}).$$

(b) Let $q : V \longrightarrow \mathbb{R}$ be a quadratic form on a real vector space V . Explain what is meant by saying that q is *positive definite*.

Determine, with a reason, whether the following two real quadratic forms are equivalent:

$$\begin{aligned} q_1(x, y, z) &= 2x^2 + y^2 + 3z^2 + 3xy + 2xz \\ q_2(x, y, z) &= x^2 + 2y^2 + z^2 + 4xy - 4yz. \end{aligned}$$

Further, determine with a reason if q_1 is positive definite.

Question 4 (a) Let $(V, \langle \cdot, \cdot \rangle)$ be a complex inner product space and let u be a vector in V satisfying $\langle u, u \rangle = 1$. Given any $v \in V$ and $\lambda \in \mathbb{C}$, show that

$$\|v - \langle v, u \rangle u\| \leq \|v - \lambda u\|.$$

In the Euclidean space $\mathbb{C}^3$, find the point u , among all points in the line

$$L = \{(x, y, 0) \in \mathbb{C}^3 : x = y\},$$

having the shortest Euclidean distance to the point $(1, 1, 1)$. Further, find this distance.

(b) Find an orthonormal basis for the subspace V of $\mathbb{R}^4$, with the standard inner product, spanned by the set $\{(1, 1, 0, 1), (0, 1, 0, 1), (-1, 0, 2, 0)\}$.

Let $\pi : \mathbb{R}^4 \longrightarrow V$ be the orthogonal projection. Find the image $\pi(1, 0, 1, 1)$ in V .

Question 5 (a) Let $\alpha : V \longrightarrow V$ be a self-adjoint linear map on a finite dimensional complex inner product space V .

Show that all eigenvalues of α are real numbers.

Let $\lambda_1, \dots, \lambda_k$ be all the distinct eigenvalues of α and let $E(\lambda_1), \dots, E(\lambda_k)$ be their eigenspaces respectively. Show that

$$V = E(\lambda_1) \oplus \dots \oplus E(\lambda_k).$$

(b) Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Find an invertible matrix P and a diagonal matrix D such that $P^T A P = I$ and $P^T B P = D$, where I is the identity matrix.

Question 6 (a) Let F be a field and let $F^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1, \dots, x_n \in F \right\}$ be the vector space of $n \times 1$ matrices over F .

Let $f : \overbrace{F^n \times \dots \times F^n}^{n\text{-times}} \longrightarrow F$ be an n -linear form. Explain what is meant by saying that f is *alternating*.

Given two non-zero alternating 3-forms f and g on F^3 , show that there is an element $\lambda \in F$ such that $f = \lambda g$.

(b) Let A and B be two $n \times n$ matrices over F . Define the *determinant* of A , $\det A$, in terms of an alternating n -form f on F^n satisfying $f(e_1, \dots, e_n) = 1$, where $\{e_1, \dots, e_n\}$ is the standard basis of F^n .

Prove the following two assertions:

(i) $\det AB = (\det A)(\det B)$;

(ii) $AB = I$ if, and only if, $BA = I$, where I is the identity matrix.

(c) Find the minimal polynomial of the matrix

$$\begin{pmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$